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31 p

Reductions of the Hypergeometric Functions

by

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1956

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ABSTRACT

The main body of this book is devoted to a collection of formulae relating the hypergeometric functions to better known functions. There is an introductory exposition of the theoretical results on hypergeometric functions. Proofs are omitted but references are supplied for those interested in the proofs.

I. THE HYPERGEOMETRIC FUNCTION

The purpose of this introduction is to give a brief summary of the theoretical aspects of the hypergeometric function. A more complete discussion may be found in [23].

The hypergeometric series:

$$(1)\quad 1 + \frac{a\,b}{1\cdot c}\,z + \frac{a\,(a+1)\,b\,(b+1)}{1\cdot 2\quad c\,(c+1)}\,z^2 + \ldots = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(n+1)\Gamma(c+n)}\,z^n$$

is absolutely convergent when $|z| < 1$ providing c is not zero or a negative integer. When $|z| = 1$ the series is absolutely convergent providing Re $(a + b - c) < 0$. The hypergeometric series thus defines a function F (a, b; c; z) which is analytic inside the unit circle. If a cut is made from +1 to $+\infty$, F (a, b; c; z) is analytic throughout the cut plane.

At the outset, one notes the almost trivial identities

$$(2)\quad F\,(a,\ b;\ c;\ z) = F\,(b,\ a;\ c;\ z)$$

$$(3)\quad \frac{d}{dz}\,F\,(a,\ b;\ c;\ z) = \frac{ab}{c}\,F\,(a+1,\ b+1;\ c+1;\ z)$$

Term by term differentiation of (1) shows that F (a, b; c; z) satisfies the differential equation:

$$(4)\quad z\,(1-z)\,\frac{d^2u}{dz^2} + \left[c - (a+b+1)\,z\right]\frac{du}{dz} - a\,b\,u = 0$$

and from (1) and (3) F (a, b; c; z) could have been defined as that solution of (4) which satisfies the initial conditions:

$$(5)\quad \begin{aligned} u\,(0) &= 1 \\ u'(0) &= \frac{ab}{c} \end{aligned}$$

Riemann observed that the hypergeometric equation (4) is completely described by its singularities. That is to say (4) has three regular singularities, one at $z = 0$, one at $z = 1$, and one at $z = \infty$, with exponents 0 and $1 - c$, 0 and $c - (a+b)$, and a and b respectively. Thus

$$(6)\qquad u = P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-(a+b) & \end{Bmatrix}$$

indicates that u satisfies the hypergeometric equation. In (6) the singularities of (4) are written in the first row and the exponents of the singularities are written beneath the appropriate singularity.

More generally the fact that u satisfies a second order linear differential equation with three regular singularities at a, b and c with exponents* α, α'; β; β'; and γ , γ' respectively is indicated by writing:

$$(7)\qquad u = P\begin{pmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{pmatrix}$$

The differential equation (called the P-equation of Riemann and Papperitz) is:

$$(8)\qquad \frac{d^2u}{dz^2} + \left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right)\frac{du}{dz}$$

$$+\left[\frac{\alpha\,\alpha'\,(a-b)\,(a-c)}{z-a} + \frac{\beta\,\beta'\,(b-c)\,(b-a)}{z-b} + \frac{\gamma\,\gamma'\,(c-a)\,(c-b)}{z-c}\right]$$

$$X\ \frac{u}{(z-a)\,(z-b)\,(z-c)} = 0$$

where the exponents must satisfy

$$(9)\qquad \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

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* It is assumed here that none of the exponent differences are integral. In this exceptional case, the complete solution may involve logarithmic terms and is discussed in [10] .

By direct transformation of (8) it may be verified that

$$(10)\qquad \left(\frac{z-a}{z-b}\right)^{k}\left(\frac{z-c}{z-b}\right)^{h} P\begin{pmatrix} a & b & c & \\ \alpha & \beta & & z \\ \alpha' & \beta' & ' & \end{pmatrix}$$

$$= P\begin{pmatrix} a & b & c & \\ \alpha+k & \beta-k-h & \gamma+h & z \\ \alpha'+k & \beta'-k-h & \gamma'+h & \end{pmatrix}$$

and

$$(11)\quad P\begin{pmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{pmatrix} = P\begin{pmatrix} a_1 & b_1 & c_1 & \\ \alpha & \beta & \gamma & z_1 \\ \alpha' & \beta' & \gamma' & \end{pmatrix}$$

(where a_1, b_1, c_1, and z_1, are derived from a, b, c, and z by the same homographic transformation

$$z_1 = \frac{az+b}{cz+d}\quad , \qquad ad - bc \neq 0).$$

$$(12)\quad P\begin{pmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{pmatrix} = \left(\frac{z-a}{z-b}\right)\left(\frac{z-c}{z-b}\right) P\begin{pmatrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+ & 0 & x \\ \alpha'-\alpha & \beta'+\alpha+ & \gamma'-\gamma & \end{pmatrix}$$

where $$x = \frac{(z-a)\ (c-b)}{(z-b)\ (c-a)}\ .$$

<u>Hence the solution of Riemann's P-equation can always be written in terms of the solution of a hypergeometric equation</u>.

From equation (8) α and α' or γ and γ' or both may be interchanged so that (12) yields four solutions in terms of the hypergeometric series. Equation (8) is also invariant under any permutation of the triplets (a, α, α'), (b, β, β') and (c, γ, γ') so that a total of 24 solutions may be obtained from (12). These 24 solutions are listed in [23] and furnish 24 solutions of the hypergeometric equation.

From the general theory of linear differential equations, any three of these solutions which have a common domain of existence must be linearly dependent in that domain. For a listing of the relations connecting the 24 solutions see [3].

The hypergeometric series may be represented by a contour integral:

$$(13) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(-s)}{\Gamma(1+s)\,\Gamma(c+s)} (-z)^s \, ds$$

where $|\arg(-z)| < \pi$ and the path of integration is such that the poles of $\Gamma(a+s)\,\Gamma(b+s)$ lie on the left of the path and the poles of $\Gamma(-s)$ lie to the right of it (it is assumed that neither a nor b is a negative integer).

The analytic continuation of the hypergeometric series can be obtained from (13) and is found to be:

$$(14) \quad \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)} (z)^{-a} F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right)$$

$$+ \frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)} (-z)^{-b} F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right)$$

where $|\arg(-z)| < \pi$ and $|z| > 1$.

When Re $(c) >$ Re $(b) > 0$, a more convenient real integral representation for the hypergeometric series is:

$$(15) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} (1-u\,z)^{-a} \, du \quad .$$

If in the hypergeometric function $F(a, b; c; z)$ one of the parameters a, b, or c is replaced by $a \pm 1$, $b \pm 1$ or $c \pm 1$ respectively, one obtains a function which is said to be contiguous to $F(a, b; c; z)$. It can be shown [23] that a linear relation exists between a hypergeometric function and any two of its contiguous functions. Thus, fifteen relations which were first found by Gauss, can be derived. Because of their usefulness in reducing hypergeometric functions to simpler functions, these relations are listed below. Writing F, $F(a \pm 1)$, $F(b \pm 1)$, and $F(c \pm 1)$ for $F(a, b; c; z)$, $F(a \pm 1, b; c; z)$, $F(a, b \pm 1; c; z)$, and $F(a, b; c \pm 1; z)$ respectively, these relations are:

$$(16) \quad [c - 2a - (b - a)z] F + a(1 - z) F(a + 1) - (c - a) F(a - 1) = 0$$

$$(17) \quad (b - a) F + a F(a + 1) - b F(b + 1) = 0$$

$$(18) \quad (c - a - b) F + a(1 - z) F(a + 1) - (c - b) F(b - 1) = 0$$

(19) $c\left[a - (c - b)\, z\right] F - ac\,(1 - z)\, F\,(a + 1) + (c - a)\,(c - b)\, z\, F\,(c + 1) = 0$

(20) $(c - a - 1)\, F + aF\,(a + 1) - (c - 1)\, F\,(c - 1) = 0$

(21) $(c - a - b)\, F - (c - a)\, F\,(a - 1) + b(1 - z)\, F\,(b + 1) = 0$

(22) $(b - a)\,(1 - z)\, F - (c - a)\, F\,(a - 1) + (c - b)\, F\,(b - 1) = 0$

(23) $c\,(1 - z)\, F - c\, F\,(a - 1) + (c - b)\, z\, F\,(c + 1) = 0$

(24) $\left[a - 1 - (c - b - 1)\, z\right] F + (c - a)\, F\,(a - 1) - (c - 1)\,(1 - z)\, F\,(c - 1) = 0$

(25) $\left[c - 2b + (b-a)\, z\right] F + b\,(1 - z)\, F\,(b + 1) - (c - b)\, F\,(b - 1) = 0$

(26) $c\left[b - (c-a)\, z\right] F - bc\,(1 - z)\, F\,(b + 1) + (c - a)(c - b)\, z\, F\,(c + 1) = 0$

(27) $(c - b - 1)\, F + b\, F\,(b + 1) - (c - 1)\, F\,(c - 1) = 0$

(28) $c\,(1 - z)\, F - c\, F\,(b - 1) + (c - a)\, z\, F\,(c + 1) = 0$

(29) $\left[b - 1 - (c-a-1)\, z\right] F + (c - b)\, F\,(b - 1) - (c - 1)(1 - z)\, F\,(c - 1) = 0$

(30) $c\left[c - 1 - (2c-a-b-1)\, z\right] F + (c - a)(c - b)\, z\, F\,(c + 1) - c(c - 1)(1 - z)\, F\,(c - 1) = 0$

II. THE CONFLUENT HYPERGEOMETRIC FUNCTION

If in a differential equation two of the singularities tend to coincidence, a differential equation results which is known as a confluent form of the original.

In particular, if in the equation satisfied by $F\ (a, b; c; \frac{z}{b})$ the limit is taken as $\underline{b}$ approaches $\underline{\infty}$ the confluent hypergeometric equation results,

$$(31) \quad z \frac{d^2 u}{d z^2} - (z - c) \frac{du}{dz} - a\, u = 0$$

one solution of (31) is the confluent hypergeometric series

$$(32) \quad {}_1F_1\ (a;\ c;\ z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)\ n!}\ z^n$$

Equation (31) has a regular singularity at zero and an irregular singularity at $\underline{\infty}$.

Since $F\ (a, b; c; z)$ is symmetric in $\underline{a}$ and $\underline{b}$, equation (15) becomes, by confluence when $\mathrm{Re}\ (c) > \mathrm{Re}\ (a) > 0$:

$$(33) \quad {}_1F_1\ (a;\ c;\ z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 e^{zu}\ u^{a-1}\ (1-u)^{c-a-1}\ du$$

Another solution to (31) is given by

$$(34) \quad \psi\ (a;\ c;\ z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zu}\ u^{a-1}\ (1+u)^{c-a-1}\ du$$

where $\mathrm{Re}\ (a) > 0$ and $\mathrm{Re}\ (z) > 0$.

Many authors prefer to treat the Whittaker functions $M_{k,m}\ (z)$, and $W_{k,m}\ (z)$ which are related to the confluent hypergeometric function by the formulae

$$(35) \quad \begin{aligned} M_{k,m}\ (z) &= e^{-z/2}\ z^{m+\frac{1}{2}}\ {}_1F_1\ (\tfrac{1}{2} - k + m;\ 2m + 1;\ z) \\ W_{k,m}\ (z) &= e^{-z/2}\ z^{m+\frac{1}{2}}\ (\tfrac{1}{2} - k + m;\ 2m + 1;\ z) \end{aligned}$$

The Whittaker functions satisfy the differential equation

$$(36) \quad \frac{d^2 w}{d z^2} + \left\{ -\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right\} w = 0$$

and are discussed in [23] and [24] .

III. REDUCTIONS OF HYPERGEOMETRIC FUNCTIONS

AND EXPLANATION OF THE TABLE

As previously noted, any second order linear differential equation with three regular singularities can be reduced to the hypergeometric equation by (12). This equation in turn may then be solved in terms of the hypergeometric function. Several ways are open for reducing the hypergeometric and confluent hypergeometric functions to better known (in the sense at least that better tabulations exist) functions. For example, one might simply examine the hypergeometric series for special parameter values. Thus, ${}_1F_1$ (a; c; z) and F (a, b; c; z) reduce to polynomials whenever a is a negative integer. Again, it may be possible to evaluate the integrals (13), (15), (18), or (19) which represent the hypergeometric function; or it may be possible to find the complete solution of the hypergeometric equation in terms of better known functions.

With these and other techniques, and with a perusal of the literature, the following tables have been collected in the hope of making the task of the engineer and physicist somewhat easier.

The organization of the results is, as far as possible, lexicographic with respect to parameter values. Since F (a, b; c; z) is symmetric with respect to a and b the smaller parameter has been chosen to be first. A brief glance at the section headings should suffice to acquaint the user with the other points of organization.

DEFINITIONS

1. Bessel Functions

$$J_\gamma(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\gamma \int_{-\infty}^{(0+)} t^{-n-1} e^{\left(t - \frac{z^2}{4t}\right)} dt$$

$$Y_\gamma(z) = \frac{J_\gamma(z)\cos\gamma\pi - J_{-\gamma}(z)}{\sin\gamma\pi}$$

2. Modified Bessel Functions

$$I_\gamma(z) = i^{-\gamma} J_\gamma(iz)$$

$$K_\gamma(z) = \frac{1}{2}\pi\left\{I_{-\gamma}(z) - I_\gamma(z)\right\}\cot\gamma\pi$$

3. Cosine and Sine Integrals

$$Ci(z) = -\int_z^\infty \frac{\cos x}{x} dx$$

$$Si(z) = \int_0^z \frac{\sin x}{x} dx$$

4. Confluent Hypergeometric Functions

$${}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)\, n!} z^n$$

$$\psi(a; c; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zu} u^{a-1} (1+u)^{c-a-1} du$$

$$Re(a) > 0,\ Re(z) > 0$$

5. **Elliptic Integrals**

$$E(k) = \int_0^{\pi/2} (1-k^2 \sin^2 \theta)^{1/2} \, d\theta$$

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} \, d\theta$$

6. **Error Functions**

$$\text{Erf}\,(x) = \int_0^{x} e^{-u^2} \, du$$

$$\text{Erfc}\,(x) = \int_x^{\infty} e^{-u^2} \, du$$

7. **Exponential Integral**

$$-\text{Ei}\,(-x) = \int_x^{\infty} \frac{e^{-u}}{u} \, du$$

8. **Fresnel Integrals**

$$C\,(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x} \frac{\cos u}{\sqrt{u}} \, du$$

$$S\,(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x} \frac{\sin u}{\sqrt{u}} \, du$$

9. **Gamma Function**

$$\Gamma(x) = \int_0^{\infty} e^{-u} u^{x-1} \, du$$

10. **Incomplete Gamma Functions**

$$\Gamma(a,x) = \int_x^{\infty} e^{-u} u^{a-1} \, du$$

$$\gamma\,(a,x) = \Gamma(a) - \Gamma(a,x)$$

11. **Gegenbauer Polynomials**

$$C_n^{\gamma}(z) = \sum_{i=0}^{n} \frac{(-1)^i \Gamma(\gamma+i)\Gamma(n+2\gamma+i)\left(\frac{1}{2}-\frac{1}{2}z\right)^i}{i!\,\Gamma(\gamma)\,\Gamma(2i+2\gamma)\,(n-i)!}$$

12. **Hankel Functions**

$$H_\gamma^{(1)}(z) = J_\gamma(z) + i\,Y_\gamma(z)$$

$$H_\gamma^{(2)}(z) = J_\gamma(z) - i\,Y_\gamma(z)$$

13. **Hypergeometric Function**

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{n!\,\Gamma(c+n)} z^n$$

14. **Legendre Functions**

$$P_\gamma^m(z) = \frac{1}{\Gamma(1-m)} \left(\frac{1+z}{1-z}\right)^{\frac{m}{2}} F\left(-\gamma, \gamma+1; 1-m; \frac{1-z}{2}\right)$$

$$\operatorname{Re}(m) < \tfrac{1}{2}; \quad |z| < 1$$

$$P_\gamma(z) = P_\gamma^0(z)$$

$$Q_\gamma^m(z) = \frac{\pi}{2\sin m\pi} \left[P_\gamma^m(z)\cos m\pi - \frac{\Gamma(\gamma+m+1)}{\Gamma(\gamma-m+1)} P_\gamma^{-m}(z) \right]$$

$$Q_\gamma(z) = Q_\gamma^0(z)$$

15. **Logarithmic Integral**

$$Li(x) = \int_0^x \frac{dt}{\log t} = Ei(\log x)$$

16. $$\psi(z) = \lim_{n \to \infty} \left[\operatorname{Log}(n\,z) - \sum_{i=0}^{n} \frac{1}{z+i} \right]$$

17. **Whittaker Functions**

$$M_{k,m}(x) = e^{-x/2}\, x^{m+\frac{1}{2}}\; {}_1F_1\left(\tfrac{1}{2} - k + m;\ 2m+1;\ x\right)$$

$$W_{k,m}(x) = e^{-x/2}\, x^{m+\frac{1}{2}}\; \psi\left(\tfrac{1}{2} - k + m;\ 2m+1;\ x\right)$$

1. NUMERICAL PARAMETERS

1.1 $$F\left(-\frac{1}{6}, \frac{1}{3}; \frac{2}{3}; x\right) = \left(\frac{1 + \sqrt{1-x}}{2}\right)^{1/3}$$

1.2 $$F\left(\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; x\right) = \left(\frac{1 + \sqrt{1-x}}{2}\right)^{1/2}$$

1.3 $$F\left(\frac{1}{3}, \frac{5}{6}; \frac{2}{3}; x\right) = \frac{1}{\sqrt{1-x}} \left(\frac{1 + \sqrt{1-x}}{2}\right)^{1/3}$$

1.4 $$F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{2}{\pi} K(k)$$

1.5 $$F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \frac{2}{\pi} E(k)$$

1.6 $$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = \frac{1}{x} \sin^{-1} x$$

1.7 $$F\left(\frac{1}{2}, 1; 1; x\right) = \frac{1}{\sqrt{1-x}}$$

1.8 $$F\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right) = \frac{1}{x} \tan^{-1} x$$

1.9 $$F\left(\frac{1}{2}, 1; \frac{3}{2}; x^2\right) = \frac{1}{2x} \ln \frac{1+x}{1-x}$$

1.10 $$F\left(\frac{1}{2}, 1; 2; x\right) = \frac{2}{1 + \sqrt{1-x}}$$

1.11 $$F\left(\frac{1}{2}, \frac{5}{4}; \frac{1}{4}; x\right) = \frac{1+x}{(1-x)^{3/2}}$$

1.12 $$F\left(\frac{2}{3}, \frac{4}{3}; \frac{1}{3}; x\right) = \frac{1+x}{(1-x)^{5/3}}$$

1.13 $$F\left(\frac{3}{4}, \frac{5}{4}; \frac{3}{2}; \frac{1}{\left[\cosh x\right]^2}\right) = \frac{1}{e^{x/2} \tanh x \, (2 \cosh x)^{1/2}}$$

1.14 $$F\left(1, 1; \frac{3}{2}; \sin^2 x\right) = \frac{x}{\sin x \cos x}$$

1.15 $$F(1, 1; 2; -x) = \frac{1}{x} \ln (1+x)$$

1.16 $$F\left(1, \frac{3}{2}; \frac{1}{2}; x\right) = \frac{1+x}{(1-x)^2}$$

1.17 $$F\left(1, \frac{3}{2}; 2; x\right) = \frac{2}{1-x+(1-x)^{1/2}}$$

1.18 $$F\left(1, \frac{3}{2}; 2; \right) \frac{1}{\left[\cosh x\right]^2} = \frac{1}{2 e^x \cosh x \tanh x}$$

1.19 $$F\left(1, \frac{3}{2}; 3; x\right) = \left[\frac{2}{1+\sqrt{1-x}}\right]^2$$

1.20 $$\lim_{\beta \to \infty} F\left(1, \beta; 1; \frac{x}{\beta}\right) = e^x$$

1.21 $$F\left(\frac{5}{6}, \frac{4}{3}; \frac{5}{3}; x\right) = \frac{1}{(1-x)^{1/2}} \left[\frac{2}{1+(1-x)^{1/2}}\right]^{2/3}$$

1.22 $$F\left(\frac{4}{3}, \frac{11}{6}; \frac{5}{3}; x\right)$$

$$= \frac{1}{5(1-x)} \left\{ \frac{6}{\sqrt{1-x}} \left[\frac{1+\sqrt{1-x}}{2}\right]^{1/3} - \left[\frac{2}{1+\sqrt{1-x}}\right]^{2/3} \right\}$$

1.23 $$F\left(\tfrac{3}{2}, \tfrac{3}{2}; \tfrac{5}{2}; \sin^2 x\right) = \frac{6}{\sin x}\left[\frac{1 - x\cot x}{\sin 2x}\right]$$

1.24 $$F\left(\tfrac{3}{2}, 2; \tfrac{1}{2}; x\right) = \frac{3x+1}{(1-x)^3}$$

1.25 $$F\left(\tfrac{3}{2}, 2; 2; x\right) = \frac{1}{(1-x)^{3/2}}$$

1.26 $$F\left(\tfrac{3}{2}, 2; \tfrac{5}{2}; -x^2\right) = \frac{3}{2}\left[\frac{(1+x^2)\tan^{-1}x - x}{x^3(1+x^2)}\right]$$

1.27 $$F\left(\tfrac{3}{2}, 2; \tfrac{5}{2}; x^2\right) = \frac{3}{4x}\left[\frac{2}{1-x^2} - \frac{1}{x^2}\ln\frac{1+x}{1-x}\right]$$

1.28 $$F\left(\tfrac{3}{2}, 2; 3; x\right) = \frac{4}{\sqrt{1-x}\,(1+\sqrt{1-x})^2}$$

1.29 $$F\left(\tfrac{3}{2}, 2; 4; x\right) = \left(\frac{2}{1+\sqrt{1-x}}\right)^3$$

1.30 $$F\left(\tfrac{5}{3}, \tfrac{7}{3}; \tfrac{4}{3}; x\right) = \frac{3}{4}\left[\frac{x+4}{(1-x)^{8/3}}\right]$$

1.31 $$F(2, 2; 1; x) = \frac{1+x}{(1-x)^3}$$

1.32 $$F\left(2, 2; \tfrac{5}{2}; \sin^2 x\right) = \frac{3}{\sin^3 2x}\left[\sin^2 2x - 2x\cos 2x\right]$$

1.33 $$F(2, 2; 3; -x) = 2\left[\frac{1}{x(1+x)} - \frac{1}{x^2}\ln(1+x)\right]$$

1.34 $$F(2, \tfrac{5}{2}; \tfrac{3}{2}; x) = \frac{1}{(1-x)^3}$$

1.35 $$F(2, \tfrac{5}{2}; 3; x) = \frac{4}{3}\left[\frac{2\sqrt{1-x}+1}{\sqrt{1-x}\,(1-x+\sqrt{1-x})^2}\right]$$

1.36 $$F(2, \tfrac{5}{2}; 4; x) = \frac{8}{\sqrt{1-x}\,(1+\sqrt{1-x})^3}$$

1.37 $$F(2, 3; 1; x) = \frac{2x+1}{(1-x)^4}$$

1.38 $$F(\tfrac{5}{2}, 3; \tfrac{3}{2}; x) = \frac{1+x}{(1-x)^4}$$

1.39 $$F(\tfrac{5}{2}, 3; 3; x) = \frac{1}{(1-x)^{5/2}}$$

1.40 $$F(3, 4; 2; x) = \frac{1+x}{(1-x)^5}$$

1.41 $$\lim_{a,b\to\infty} F(a, b; \tfrac{1}{2}; -\tfrac{x^2}{4ab}) = \cos x$$

1.42 $$\lim_{a,b\to\infty} F(a, b; \tfrac{1}{2}; \tfrac{x^2}{4ab}) = \cosh x$$

1.43 $$\lim_{a,b\to\infty} F(a, b; \tfrac{3}{2}; -\tfrac{x^2}{4ab}) = \frac{1}{x}\sin x$$

1.44 $$\lim_{a,b\to\infty} F(a, b; \tfrac{3}{2}; \tfrac{x^2}{4ab}) = \frac{1}{x}\sinh x$$

2. FIRST AND THIRD PARAMETERS NUMERICAL

2.1 $$\lim_{b \to 0} F\,(b,\ -n\ ;\ 2b\ ;\ -x) = \frac{1 + (1+x)^n}{2}$$

2.2 $$F\,(1\ ,\ 1-n\ ;\ 2\ ;\ -x) = \frac{(1+x)^n - 1}{nx}$$

3. FIRST PARAMETER NUMERICAL

3.1 $$F\left(\frac{1}{2}, -n; \frac{1}{2} - n; x^2\right) = \frac{2^{2n}\left[\Gamma(n+1)\right]^2}{\Gamma(2n+1)} P_n\left(\frac{1+x^2}{2x}\right)$$

3.2 $$F\left(\frac{1}{2}, n+1; n+\frac{3}{2}; x^2\right)$$

$$= \frac{\Gamma(2n+2)}{2^{2n+1}\left[\Gamma(n+1)\right]^2 x^{n+1}} Q_n\left(\frac{1+x^2}{2x}\right)$$

3.3 $$F(-1, \beta; \beta; -x) = 1 + x$$

3.4 $$F\left(1, -m; a-m+1; -\frac{1}{x}\right) = \frac{1}{x^m \binom{a}{m}} \sum_{i=0}^{m} \binom{a}{i} x^i$$

3.5 $$F(1, m+1-a; m+2; -x)$$

$$= \frac{\Gamma(a-m)(m+1)!}{\Gamma(a+1)\, x^{m+1}} \sum_{n=m+1}^{\infty} \binom{a}{n} x^n$$

3.6 $$F(-2, \beta; \beta; -x) = (1+x)^2$$

4. THIRD PARAMETER NUMERICAL

4.1 $$F\left(\frac{1}{2} n , - \frac{1}{2} n ; \frac{1}{2} ; \sin^2 x\right) = \cos n x$$

4.2 $$F\left(\frac{1}{2} n , \frac{1}{2} n + \frac{1}{2} ; \frac{1}{2} ; - \tan^2 x\right) = \cos n x \cos^n x$$

4.3 $$F\left(- \frac{1}{2} n , - \frac{1}{2} n + \frac{1}{2} ; \frac{1}{2} ; x^2\right) = \frac{(1+x)^n + (1-x)^n}{2}$$

4.4 $$F\left(-\frac{1}{2} a , \frac{1}{2} - \frac{1}{2} a ; \frac{1}{2} ; - \tan^2 x\right) = \frac{\cos a x}{\cos^a x}$$

4.5 $$F\left(a , a + \frac{1}{2} ; \frac{1}{2} ; x\right) = \frac{1}{2} (1 + x^{1/2})^{-2a} + \frac{1}{2} (1 - x^{1/2})^{-2a}$$

4.6 $$F\left(- n , n + \frac{1}{2} ; \frac{1}{2} ; x^2\right) = \frac{(-1)^n \, 2^{2n} \, (n!)^2}{(2n)!} P_{2n} (x)$$

4.7 $$F\left(\frac{1}{2} - \frac{1}{2} a , \frac{1}{2} + \frac{1}{2} a ; \frac{1}{2} ; \sin^2 x\right) = \frac{\cos a x}{\cos x}$$

4.8 $$F\left(\frac{\gamma+1}{2} , - \frac{\gamma}{2} ; 1 ; 1 - x^2\right) = P_\gamma (x)$$

4.9 $$F\left(1 + \gamma , - \gamma ; 1 ; \frac{1}{2} - \frac{1}{2} x\right) = P_\gamma (x)$$

4.10 $$F\left(- n , - n ; 1 ; \tan^2 \frac{1}{2} x\right) = \frac{1}{\cos^n \frac{1}{2} x} P_n (\cos x)$$

4.11 $$F\left(n + 1 , - n ; 1 ; \cos^2 \frac{1}{2} x\right) = (-1)^n P_n (\cos x)$$

4.12 $$F\left(\frac{1}{2} - \frac{1}{2} n , 1 - \frac{1}{2} n ; \frac{3}{2} ; x^2\right) = \frac{(1 + x)^n - (1 - x)^n}{2 n x}$$

4.13 $$F\left(\frac{1}{2} - \frac{1}{2} n , 1 - \frac{1}{2} n ; \frac{3}{2} ; - \tan^2 x\right) = \frac{\sin n x}{n \sin x \cos^{n-1} x}$$

4.14 $$F\left(\frac{1}{2} - \frac{1}{2} a , \frac{1}{2} + \frac{1}{2} a ; \frac{3}{2} ; \sin^2 x\right) = \frac{\sin a x}{a \sin x}$$

4.15 $$F\left(1 - \frac{1}{2} a , 1 + \frac{1}{2} a ; \frac{3}{2} ; \sin^2 x\right) = \frac{\sin a x}{a \sin x \cos x}$$

4.16 $$F\left(\frac{1}{2} + \frac{1}{2} n , 1 + \frac{1}{2} n ; \frac{3}{2} ; - \tan^2 x\right) = \frac{\sin n x \cos^{n+1} x}{n \sin x}$$

4.17 $$F\left(- n , \frac{3}{2} + n ; \frac{3}{2} ; x^2\right) = \frac{(- 1)^n (n!)^2 2^{2n}}{(2 n + 1) ! x} P_{2n+1} (x)$$

5. NO NUMERICAL PARAMETERS

5.1 $$F\,(-n\,,\,\beta\,;\,\beta\,;\,-x) = (1+x)^{n}$$

5.2 $$F\,(a-\tfrac{1}{2}\,,\,a\,;\,2a\,;\,x) = \left[\frac{1}{2} + \frac{(1-x)^{1/2}}{2}\right]^{1-2a}$$

5.3 $$F\,(a\,,\,a+\tfrac{1}{2}\,;\,2a\,;\,x) = \frac{1}{(1-x)^{1/2}}\left[\frac{1}{2} + \frac{(1-x)^{1/2}}{2}\right]^{1-2a}$$

5.4 $$F\,(a\,,\,1+\tfrac{1}{2}a\,;\,\tfrac{1}{2}a\,;\,x) = \frac{1+x}{(1-x)^{a+1}}$$

5.5 $$F\,(\tfrac{1}{2}n\,,\,n-1\,;\,\tfrac{1}{2}n-1\,;\,x) = \frac{n\,x+n-2}{(n-2)\,(1-x)^{n}}$$

5.6 $$F\left(1+\tfrac{1}{2}a\,,\,\tfrac{1}{2}+\tfrac{1}{2}a\,;\,1+a\,;\,\frac{1}{\left[\cosh x\right]^{2}}\right) = \frac{e^{-ax}\,(2\cosh x)^{-a}}{\tanh x}$$

5.7 $$F\,(\frac{\gamma-\mu+1}{2}\,,\,-\frac{\gamma+\mu}{2}\,;\,1-\mu\,;\,1-x^{2})$$

$$= \frac{(x^{2}-1)^{\mu}\,\Gamma(1-\mu)}{2^{\mu}}\quad P^{\mu}_{\gamma}\;(x)$$

5.8 $$F\,(-\gamma,\,\gamma+1\,;\,1-\mu\,;\,\tfrac{1}{2}-\tfrac{1}{2}x)$$

$$= \Gamma(1-\mu)\left(\frac{x-1}{x+1}\right)^{1/2\mu}\quad P^{\mu}_{\gamma}\;(x)$$

5.9 $$F\,(\tfrac{1}{2}\gamma+\tfrac{1}{2}\mu+1\,,\,\tfrac{1}{2}\gamma+\tfrac{1}{2}\mu+\tfrac{1}{2}\,;\,\gamma+\tfrac{3}{2}\,;\,\frac{1}{x^{2}})$$

$$= \frac{e^{-\mu i\pi}\,2^{\gamma+1}\,\Gamma(\gamma+\tfrac{3}{2})\,x^{\gamma+\mu+1}}{\pi^{1/2}\,\Gamma(\gamma+\mu+1)\,(x^{2}-1)^{\mu/2}}\quad Q^{\mu}_{\gamma}\;(x)$$

5.10 $$F\left(1-\mu+\gamma,\ -\mu-\gamma;\ 1-\mu;\ \tfrac{1}{2}-\tfrac{1}{2}x\right) = \frac{\Gamma(1-\mu)\,(x^2-1)^{\frac{1}{2}\mu}}{2^{\mu}}\ P_{\gamma}^{\mu}(x)$$

5.11 $$F\left(\tfrac{1}{2}+\tfrac{1}{2}n,\ 1+\tfrac{1}{2}n;\ \tfrac{3}{2}+n;\ \frac{1}{x^2}\right) = \frac{(2x)^{n+1}\,\Gamma(n+\frac{3}{2})}{\pi^{1/2}\,\Gamma(n+1)}\ Q_n(x)$$

5.12 $$F\left(\tfrac{1}{2}+\tfrac{1}{2}\gamma-\tfrac{1}{2}\mu,\ 1+\tfrac{1}{2}\gamma-\tfrac{1}{2}\mu;\ \gamma+\tfrac{3}{2};\ \frac{1}{x^2}\right) = \frac{e^{-\mu i\pi}\,2^{\gamma+1}\,\Gamma(\gamma+\frac{3}{2})\,x^{1+\gamma-\mu}\,(x^2-1)^{\frac{1}{2}\mu}}{\pi^{\frac{1}{2}}\,\Gamma(1+\gamma+\mu)}\ Q_{\gamma}^{\mu}(x)$$

5.13 $$F\left(2\gamma+n,\ -n;\ \gamma+\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}x\right) = \frac{\Gamma(n+1)\,\Gamma(2\gamma)}{\Gamma(2\gamma+n)}\ C_n^{\gamma}(x)$$

5.14 $$F\left(-\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}n;\ 1-\gamma-n;\ \frac{1}{x^2}\right) = \frac{n!\,\Gamma(\gamma)}{(2x)^n\,\Gamma(\gamma+n)}\ C_n^{\gamma}(x)$$

5.15 $$F\left(\frac{\mu+\gamma+1}{2},\ \frac{\mu-\gamma}{2};\ 1+\mu;\ \frac{r^2}{x^2+r^2}\right) = \frac{2^{\mu}\,(x^2+r^2)^{\frac{\mu+\gamma+1}{2}}\,\Gamma(\mu+1)\,\Gamma(\gamma-\mu+1)}{r^{\mu}\,\Gamma(\mu+\gamma+1)\,R^{\gamma+1}}\ P_{\gamma}^{\mu}(\cos\theta)$$

where

$$x = R\cos\theta\ ,\quad r = R\sin\theta$$

5.16 $$F\left(\frac{1}{2}+\mu,\ \frac{1}{2}-\mu;\ \frac{3}{2}+\gamma;\ \frac{1}{1-e^{2x}}\right)$$

$$= \frac{e^{-\mu i\pi}\,\Gamma(\gamma+\frac{3}{2})\,e^{(\mu+\gamma+1)x}\,(1-e^{-2x})^{\mu+\frac{1}{2}}}{2^{3}\,\pi^{1/2}\,\Gamma(\mu+\gamma+1)\,\sinh^{\mu} x}\quad Q^{\mu}_{\gamma}\,(\cosh x)$$

where $e^{2x} > 2$

5.17 $$\lim_{n\to\infty} F\left(m-n,\ m+n+1;\ m+1;\ \frac{1}{4}\left(\frac{x}{n}\right)^{2}\right)$$

$$= \left(\frac{2}{x}\right)^{m} m!\ J_{m}\,(x)$$

6. SPECIAL ARGUMENT VALUES

6.1 $$F\left(3a\ ,\ 3a+\frac{1}{2}\ ;\ 2a+\frac{5}{6}\ ;\ \frac{1}{9}\right) = \left(\frac{3}{4}\right)^{3a} \frac{\Gamma\left(2a+\frac{5}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(a+\frac{5}{6}\right)}$$

6.2 $$F\left(-a\ ,\ -a+\frac{1}{2}\ ;\ 2a+\frac{3}{2}\ ;\ -\frac{1}{3}\right) = \left(\frac{8}{9}\right)^{2a} \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(2a+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(2a+\frac{4}{3}\right)}$$

6.3 $$F\left(\frac{1}{2}\ ,\ \frac{1}{2}\ ;\ 1\ ;\ \frac{1}{2}\right) = \frac{1}{2\pi^{3/2}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

6.4 $$F\left(a\ ,\ 1-a\ ;\ c\ ;\ \frac{1}{2}\right)$$
$$= \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}c\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}c-\frac{1}{2}a\right)}$$

6.5 $$F\left(a\ ,\ b\ ;\ \frac{1}{2}(a+b+1)\ ;\ \frac{1}{2}\right)$$
$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b\right)}$$

6.6 $$F\left(2a\ ,\ 2b\ ;\ a+b+1\ ;\ \frac{1}{2}\right)$$
$$= \frac{\pi}{a-b}\,\Gamma(a+b+1)\left\{\frac{1}{\Gamma(a)\Gamma\left(\frac{1}{2}+b\right)} - \frac{1}{\Gamma(b)\Gamma\left(\frac{1}{2}+a\right)}\right\}$$

6.7 $$F(a\ ,\ b\ ;\ c\ ;\ 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$\mathrm{Re}\,(c) > \mathrm{Re}\,(a+b)$$

6.8 $$F(a\ ,\ b\ ;\ 1+a-b\ ;\ -1)$$
$$= \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)}$$

6.9 $$(a+1)\ F\ (-a\ ,\ 1\ ;\ b+2\ ;\ -1) + (b+1)\ F\ (-b\ ,\ 1\ ;\ a+2\ ;\ -1)$$

$$= 2^{a+b+1}\ \frac{\Gamma(a+2)\,\Gamma(b+2)}{\Gamma(a+b+2)}$$

6.10 $$F\ (1\ ,\ a\ ;\ a+1\ ;\ -1)$$

$$= 2a \left[\psi\left(\frac{1}{2} + \frac{1}{2}a\right) - \psi\left(\frac{1}{2}a\right) \right]$$

6.11 $$F\ \left(a + \frac{1}{3}\ ,\ 3a\ ;\ 2a + \frac{2}{3}\ ;\ e^{\pm i\pi/3}\right)$$

$$= \frac{2\pi\, e^{\pm i\pi a/2}}{3^{\frac{3a+1}{2}}}\ \frac{\Gamma\left(2a + \frac{2}{3}\right)}{\Gamma\left(a + \frac{1}{3}\right)\Gamma\left(a + \frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}$$

7. THE CONFLUENT HYPERGEOMETRIC FUNCTION

7.1 $${}_1F_1\left(\frac{1}{2} ; \frac{3}{2} ; -x^2\right) = \frac{\mathrm{Erf}\ (x)}{x}$$

7.2 $${}_1F_1\ (a\ ,\ a\ ;\ x) = e^x$$

7.3 $${}_1F_1\ (a\ ;\ a+1\ ;\ -x) = \frac{a}{x}\ \gamma\,(a,x)$$

7.4 $${}_1F_1\left(\frac{1}{2} + \gamma ;\ 1 + 2\gamma ;\ 2\,i\,x\right)$$

$$= \Gamma(\gamma + 1)\ e^{ix} \left(\frac{1}{2}\, x\right)^{-\gamma}\ J_\gamma\ (x)$$

7.5 $${}_1F_1\left(\frac{1}{2} + \gamma ;\ 1 + 2\gamma ;\ 2x\right) = \Gamma(\gamma + 1)\ e^x \left(\frac{1}{2}\, x\right)^{-\gamma} I_\gamma\ (x)$$

7.6 $$\lim_{a \to \infty}\ {}_1F_1\left(a\ ;\ n\ ;\ -\frac{x}{a}\right) = \Gamma(n)\ J_{n-1}\ (2\, x^{1/2})$$

7.7 $${}_1F_1\left(\frac{1}{2} ; \frac{3}{2} ; -x\, e^{i\,\pi/2}\right) + {}_1F_1\left(\frac{1}{2} ; \frac{3}{2} ; -x\, e^{-i\,\pi/2}\right)$$

$$= \sqrt{\frac{2\pi}{x}}\quad C\ (x)$$

7.8 $${}_1F_1\left(\frac{1}{2} ; \frac{3}{2} ; -\, x\, e^{i\,\pi/2}\right) - {}_1F_1\left(\frac{1}{2} ; \frac{3}{2} ; -x\, e^{-i\,\pi/2}\right)$$

$$= -i\ \sqrt{\frac{2\pi}{x}}\quad S\ (x)$$

7.9 $$\psi\left(\frac{1}{2} ; \frac{1}{2} ; x^2\right) = e^{x^2}\ \mathrm{Erfc}\ x$$

7.10 $$\psi\,(1\ ;\ 1\ ;\ x) = -e^x\ \mathrm{Ei}\ (-x)$$

7.11 $$\psi(1\,;\,1\,;\,-\log x) = -\frac{\mathrm{Li}(x)}{x}$$

7.12 $$\psi(1-a\,;\,1-a\,;\,x) = e^{x}\,\Gamma(a,x)$$

7.13 $$\psi\left(\tfrac{1}{2}+\gamma;\,1+2\gamma;\,2x\right) = \pi^{-1/2}\,e^{x}\,(2x)^{-\gamma}\,K_{\gamma}(x)$$

7.14 $$\psi\left(\tfrac{1}{2}+\gamma;\,1+2\gamma;\,-2ix\right) = \tfrac{1}{2}\,\pi^{-1/2}\,e^{-i(x-\gamma\pi)}\,(2x)^{-\gamma}\,H^{(1)}_{\gamma}(x)$$

7.15 $$\psi\left(\tfrac{1}{2}+\gamma;\,1+2\gamma;\,2ix\right) = -\tfrac{1}{2}\,\pi^{-1/2}\,e^{i(x-\gamma\pi)}\,(2x)^{-\gamma}\,H^{(2)}_{\gamma}(x)$$

7.16 $$e^{i(\gamma\pi-x)}\,\psi\left(\tfrac{1}{2}+\gamma;\,1+2\gamma;\,2ix\right) + e^{i(x-\gamma\pi)}\,\psi\left(\tfrac{1}{2}+\gamma;\,1+2\gamma;\,-2ix\right)$$
$$= -\pi^{-1/2}\,(2x)^{-\gamma}\,Y_{\gamma}(x)$$

7.17 $$\tfrac{1}{2}\pi - \tfrac{1}{2}\,i\,e^{-ix}\,\psi(1\,;\,1\,;\,ix) + \tfrac{1}{2}\,i\,e^{ix}\,\psi(1\,;\,1\,;\,-ix) = \mathrm{Si}(x)$$

7.18 $$-\tfrac{1}{2}\,e^{-ix}\,\psi(1\,;\,1\,;\,ix) - \tfrac{1}{2}\,e^{ix}\,\psi(1\,;\,1\,;\,-ix) = \mathrm{Ci}(x)$$

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$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha\ (\alpha+1)\ \beta\ (\beta+1)}{1 \cdot 2 \cdot \gamma\ (\gamma+1)} x^2 + \dots$$

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